Four-Body Trajectory Optimization

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A comprehensive computer program written in Fortran has been developed to compute 2-impulse and 3-impulse fuel optimal trajectories in the sun-Earth-moon space. The program is intended for mission planning studies where the Earth, sun, and moon can all have a significant influence on the transfer trajectory. The approach is to use the Stumpff-Weiss method of state propagation with analytic calculation of the state transition matrix and Davidon's variance algorithm in an accelerated gradient procedure. A variable step integration technique and a quadrature formula for correcting single step errors are used with the Stumpff-Weiss method to improve efficiency and accuracy. The 2-impulse optimization problem minimizes the terminal cost of a transfer from an Earth parking orbit of known inclination to a given final position and velocity in fixed time. It is solved by an accelerated gradient projection formulation. The 3-impulse problem minimizes the total cost of a transfer between two fixed points in fixed time. It uses an accelerated gradient formulation to optimize cost in the outer loop and a Newton-Raphson method to solve Lambert's problem in the inner loop. The independent variables are chosen so that the number of inner loop Lambert problems to be solved is reduced to one from the classical two. Numerical examples of typical lunar-swingby trajectories between the Earth and the L_1 libration point are presented.

Introduction

A COMPREHENSIVE computer program has been developed to compute fuel optimal 4-body trajectories between the Earth and some point in the sun-Earth-moon space. The approach is to use the Stumpff-Weiss method to extrapolate the state vector and compute the state transition matrix with Davidon's variance algorithm in an accelerated gradient procedure. An important feature is that the cost and constraint gradients can be computed analytically in terms of the terminal state and the state transition matrix. The efficiency and accuracy of the integration process has been improved by incorporating a variable step technique and a quadrature formula for correcting single step errors.

The generation of a 4-body fuel optimal trajectory is considerably more difficult and time consuming than in the 2-body case mainly because of the increased difficulty in solving the Lambert problem. In a 4-body 2-impulse transfer it is generally necessary to go through a search process to obtain an initial estimate of the required velocity. If the initial estimate is reasonably good so that the terminal miss is small, an iterative solution of the boundary value problem will converge to the required velocity. In general, the solution of the Lambert problem will require several iterations, each involving a costly function evaluation (the extrapolation of the state vector and the computation of the state transition matrix). The Davidon's variance algorithm was chosen in the accelerated gradient procedure to reduce the number of function evaluations otherwise required in a one-dimensional search.[1,2] The generation of a 4-body optimal trajectory is also more difficult than the generation of 3-body optimal trajectory.[3] This is because of the additional swingby possibilities and because the motion of the primaries is a 3-body motion and cannot be found analytically.

Some reduction in computer time is possible by a judicious choice of the independent variables. For instance, the classical choice of the independent variables to optimize ΔV in a 2-body 3-impulse transfer is the position and time of the interior impulse $(\mathbf{R}_m, t_m)$. The cost gradient may be expressed in terms of the time derivative of the primer vector. Since the solution of the 2-body Lambert problem is a single step process, there is no inner loop of importance. When this approach is applied to a 4-body problem, it would require the solution of two difficult inner loop Lambert problems to satisfy constraints at two places and an outer loop to optimize cost. The 4-body problem is highly nonlinear in that the inner loops will fail to converge unless the changes in the interior impulse position and time as generated by the outer loop are heavily constrained. As a result, the progress to a converged solution tends to be very slow.

In view of the fact that the reference and the perturbed trajectories are required to satisfy the same boundary conditions, there are only 4 degrees of freedom. Thus, a better approach is to iterate on the initial required velocity and the time of the interior impulse in the outer loop. The effect of this change is that one of the two inner loop Lambert problems is eliminated. The cost gradient may be computed without computing the primer vector. This new formulation results in a significant saving in computer time. To insure convergence the required change in the interior impulse position with respect to the reference trajectory for the remaining Lambert problem is introduced in increments rather than in one single step. After the problem has converged to a solution, the primer vector history is computed to determine whether the trajectory is optimal or an additional impulse is required.

State Extrapolation

There are only two methods which are suitable for use in a 4-body trajectory optimization program. They are both multiconic methods. The first version is due to Wilson[4] and to Byrnes and Hooper,[5] and the second version due to Stumpff-Weiss.[6,7] The major difference between the two methods is that during each time step the former computes the 2-body conics in sequence while the latter computes them in a parallel manner. In a 4-body space the Stumpff-Weiss method does not require logic to determine the proper sequence of computation of the conics. While this is not a problem in a 3-body space, it introduces complications in a 4-body space. Furthermore, D'Amario and Edelbaum[8] show that all multiconic methods have similar error propagation for small steps.

Another advantage of the Stumpff-Weiss method is that it generates its own ephemerides. If the ephemerides are read, say, from a JPL tape, it is not possible to exclude the presence of the other bodies. The state extrapolation procedure at each time step is as follows. 1) Compute step size and estimates of position errors. 2) Compute 6 conics, 3 2-body state transition matrices, and 3 perturbation vectors to obtain 6 reference trajectories and the 4-body state transition matrix. 3) Compute correction of position and velocity errors by quadrature formulas to update the reference trajectories.

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Variable Step Integration

The 6 conics are computed using Goodyear's routine[9] and are denoted by

$$[\mathbf{p}_{i,j}] = ([\mathbf{R}_{i,j}], [\mathbf{V}_{i,j}]) \tag{1}$$

the 6 reference trajectories are represented by

$$\begin{gathered}\mathbf{p}_{i,j} = (\mathbf{R}_{i,j}, \mathbf{V}_{i,j}) \\ (i,j) = (SV, SE, SM, EV, MV, EM)\end{gathered} \tag{2}$$

In the above equations, $\mathbf{R}$ is position and $\mathbf{V}$ is velocity. The subscripts S, E, M, and V represents the sun, Earth, moon, and spacecraft respectively.

The estimate of the position error at the end of a single step, $\varepsilon(t+h)$, is the difference between the true position and the approximate position determined by Stumpff-Weiss method. If this estimate is expanded in Taylor series at the beginning of the step, it can be shown that the first nonzero term is the 4th derivative term. Neglecting all higher derivative terms the estimate of the position error is given by

$$\varepsilon(t+h) \approx \ddddot{\varepsilon}(t)h^4/24 \tag{3}$$

The 4th derivative terms needed are $\ddddot{\varepsilon}_{SV}$, $\ddddot{\varepsilon}_{SE}$ and $\ddddot{\varepsilon}_{SM}$.

An estimate of the step size may be computed by

$$h = (24\varepsilon_{\max}/|\ddddot{\varepsilon}(t)_{SV}|)^{1/4} \tag{4}$$

where $\varepsilon_{\max}$ = magnitude of the allowable single step position error. Let

$$\begin{gathered}G_{i,j} = \partial/\partial R_{i,j}[\mathbf{R}_{i,j}] \\ \boldsymbol{\delta}\mathbf{g}_{i,j} = \ddot{\mathbf{R}}_{i,j} - [\ddot{\mathbf{R}}_{i,j}] \\ (i,j) = (SV, EV, SE, MV, SM)\end{gathered} \tag{5}$$

the 4th derivatives of the estimated position errors can be expressed by

$$\begin{aligned}\ddddot{\varepsilon}_{SV}(t) &= G_{SV}\,\boldsymbol{\delta}\mathbf{g}_{SV} + G_{EV}\,\boldsymbol{\delta}\mathbf{g}_{EV} + \frac{\mu_E}{\mu_S+\mu_E}G_{SE}\,\boldsymbol{\delta}\mathbf{g}_{SE} + \\ &\qquad G_{MV}\,\boldsymbol{\delta}\mathbf{g}_{MV} + \frac{\mu_M}{\mu_S+\mu_M}G_{SM}\,\boldsymbol{\delta}\mathbf{g}_{SM} \\ \ddddot{\varepsilon}_{SE}(t) &= G_{SE}\,\boldsymbol{\delta}\mathbf{g}_{SE} - \frac{\mu_M}{\mu_E+\mu_M}G_{EM}\,\boldsymbol{\delta}\mathbf{g}_{EM} + \frac{\mu_M}{\mu_S+\mu_M}G_{SM}\,\boldsymbol{\delta}\mathbf{g}_{SM} \\ \ddddot{\varepsilon}_{SM}(t) &= G_{SM}\,\boldsymbol{\delta}\mathbf{g}_{SM} + \frac{\mu_E}{\mu_E+\mu_M}G_{EM}\,\boldsymbol{\delta}\mathbf{g}_{EM} + \frac{\mu_E}{\mu_S+\mu_E}G_{SE}\,\boldsymbol{\delta}\mathbf{g}_{SE}\end{aligned} \tag{6}$$

These formulas are generalizations of the three-body formulas given by D'Amario and Edelbaum.[8] Let

$$\begin{gathered}d\mathbf{P}_{i,j} = [\mathbf{p}_{i,j}] - J\mathbf{p}_{i,j} \\ J = \begin{pmatrix} I_3 & hI_3 \\ O_3 & I_3 \end{pmatrix} \\ (i,j) = (SV, SE, SM, EV, MV, EM)\end{gathered} \tag{7}$$

The 3 perturbation vectors are given by the equations below

$$\begin{aligned}\mathbf{P}_{SV} &= \frac{\mu_E}{\mu_S+\mu_E}d\mathbf{P}_{SE} + d\mathbf{P}_{EV} + \frac{\mu_M}{\mu_S+\mu_M}d\mathbf{P}_{SM} + d\mathbf{P}_{MV} \\ \mathbf{P}_{SE} &= \frac{\mu_M}{\mu_S+\mu_M}d\mathbf{P}_{SM} - \frac{\mu_M}{\mu_E+\mu_M}d\mathbf{P}_{EM} \\ \mathbf{P}_{SM} &= \frac{\mu_E}{\mu_S+\mu_E}d\mathbf{P}_{SE} + \frac{\mu_E}{\mu_E+\mu_M}d\mathbf{P}_{EM}\end{aligned} \tag{8}$$

The 6 reference trajectories before correction are

$$\begin{aligned}\mathbf{p}_{SV} &= [\mathbf{p}_{SV}] + \mathbf{P}_{SV} \\ \mathbf{p}_{SE} &= [\mathbf{p}_{SE}] + \mathbf{P}_{SE} \\ \mathbf{p}_{SM} &= [\mathbf{p}_{SM}] + \mathbf{P}_{SM} \\ \mathbf{p}_{EV} &= -\mathbf{p}_{SE} + \mathbf{p}_{SV} \\ \mathbf{p}_{MV} &= -\mathbf{p}_{SM} + \mathbf{p}_{SV} \\ \mathbf{p}_{EM} &= -\mathbf{p}_{SE} + \mathbf{p}_{SM}\end{aligned} \tag{9}$$

Correction of Errors by Quadrature Formulas

The quadrature formulas used to correct the reference trajectories are based on the assumption that the second derivative of the position error in a single step is given by

$$\ddot{\varepsilon} = t^2(\mathbf{a} + \mathbf{b}t)/2 \tag{10}$$

where $\mathbf{a} = \ddddot{\varepsilon}_0$ at beginning of a step.

Then, the coefficient $\mathbf{b}$ may be expressed in terms of the second derivative of the position error at the end of a single step

$$\mathbf{b} = \frac{2\ddot{\varepsilon}_1}{h^3} - \frac{\ddddot{\varepsilon}_0}{h} \tag{11}$$

Integrating Eq. (10), we obtain

$$\begin{aligned}\dot{\varepsilon} &= \mathbf{a}\frac{t^3}{6} + \mathbf{b}\frac{t^4}{8} \\ \varepsilon &= \mathbf{a}\frac{t^4}{24} + \mathbf{b}\frac{t^5}{40}\end{aligned} \tag{12}$$

The corrections at the end of a single step are then

$$\begin{aligned}\dot{\varepsilon}_1 &= \frac{h}{4}\left(\ddddot{\varepsilon}_0\frac{h^2}{6} + \ddot{\varepsilon}_1\right) \\ \varepsilon_1 &= \frac{h^2}{20}\left(\ddddot{\varepsilon}_0\frac{h^2}{3} + \ddot{\varepsilon}_1\right)\end{aligned} \tag{13}$$

where $\ddddot{\varepsilon}_0$ is the 4th derivative of position error estimate at the beginning of a step and $\ddot{\varepsilon}_1$ is the 2nd derivative of the Stumpff-Weiss error estimate at the end of the step.

Let

$$\begin{gathered}\mathbf{s}_{i,j} = \frac{[\mathbf{R}_{i,j}]}{|[\mathbf{R}_{i,j}]|^3} - \frac{\mathbf{R}_{i,j}}{|\mathbf{R}_{i,j}|^3} \\ (i,j) = SV, EV, MV, SE, SM, EM)\end{gathered} \tag{14}$$

The 2nd derivatives of the Stumpff-Weiss position errors are given by[6]

$$\begin{aligned}\ddot{\varepsilon}_{1,SV} &= \mu_S\mu_M\,\mathbf{s}_{SV} + \mu_E(\mathbf{s}_{SE} + \mathbf{s}_{EV}) + \mu_M(\mathbf{s}_{SM} + \mathbf{s}_{MV}) \\ \ddot{\varepsilon}_{1,SE} &= (\mu_S + \mu_E)\mathbf{s}_{SE} + \mu_M(\mathbf{s}_{SM} - \mathbf{s}_{EM}) \\ \ddot{\varepsilon}_{1,SM} &= (\mu_S + \mu_M)\mathbf{s}_{SM} + \mu_E(\mathbf{s}_{SE} + \mathbf{s}_{EM})\end{aligned} \tag{15}$$

The uncorrected reference trajectories in Eqs. (9) and (14) are used to compute the 3 2nd derivatives of the Stumpff-Weiss position errors in Eq. (15). They and the 4th derivatives of the estimated errors in Eq. (6) are used to compute corrections by Eq. (13) to update 3 reference trajectories $\mathbf{p}_{SV}$, $\mathbf{p}_{SE}$ and $\mathbf{p}_{SM}$. The latter are used to update the 3 remaining reference trajectories $\mathbf{p}_{EV}$, $\mathbf{p}_{MV}$ and $\mathbf{p}_{EM}$.

Two-Impulse Transfer

The 2-impulse optimization program minimizes the terminal cost of a transfer from an Earth parking orbit of given inclination to a given final position and velocity in fixed time. The initial cost at Earth departure is considered to be a part of the booster and is therefore not included. The velocity vector leaving the parking orbit is assumed to be normal to the position vector. The magnitude of the departure velocity, the longitude of the ascending node and the angle of the spacecraft measured from the ascending node in the orbital plane are used as the independent variables. In an early version of the program the inclination of the parking orbit was not specified and the independent variables were the magnitude of the departure velocity and three angles which define the departure position and velocity vectors. An example of the target vector is the position vector of the L_1 libration point or a point in a Halo orbit about the libration point.

The 2-impulse optimization problem is solved by a gradient projection method[10,13] using Davidon's variance algorithm. It minimizes an augmented cost function defined by

$$\mathbf{F}(x) = \mathbf{f}(x) + v^T\boldsymbol{\psi}(x) \tag{16}$$

where $f(x)$ is the cost, $\psi(x)$ is the constant violation and ν is the vector of Lagrange multiples given by

$$\nu = -[L^T V L]^{-1} L^T V \mathbf{g}$$
$$L = \partial\psi^T/\partial \mathbf{x} \quad \mathbf{g} = \partial \mathbf{f}^T/\partial \mathbf{x} \tag{17}$$

and V is the variance matrix.

A constraint restoration step is taken after an accelerated gradient step if the constraint violation has exceeded the allowable limit. The change on the independent variable in the constraint restoration step is computed by

$$d\mathbf{x} = -L(L^T L)^{-1}\psi \tag{18}$$

Three-Impulse Transfer

The 3-impulse transfer is between 2 fixed points in space in fixed time. The optimization problem is to minimize the total cost. In the classical approach there is an outer loop which iterates on the position vector and time of the interior impulse ($\mathbf{R}_m$, t_m) to reduce cost and two inner loops which iterate on the initial velocity ($\mathbf{V}_0{}^+$) and interior velocity ($\mathbf{V}_m{}^+$) to satisfy interior and terminal position constraints.[11,12]

In the new formulation the independent variables in the outer loop are changed to $\mathbf{V}_0{}^+$ and t_m. The state vector is extrapolated to t_m. There is only one inner loop which iterates on $\mathbf{V}_m{}^+$ to satisfy the terminal constraint. Since the extrapolation of state and the computation of the state transition matrix is very time consuming, this change has effected a substantial reduction of computer time.

The outer loop uses the Davidon's variance algorithm to reduce cost and the inner loop uses the Newton-Raphson method to satisfy the terminal constraint. The change in the interior impulse position due to a change in the initial velocity is usually so large that the $\mathbf{V}_m{}^+$ determined on the reference trajectory is a poor estimate for the inner loop Lambert problem of the perturbed trajectory. This difficulty is alleviated by introducing the change of interior impulse position in increments rather than in one step.

Example 1

A spacecraft is assumed to be launched in the month of July 1978 from a 100 naut mile Earth parking orbit to the L_1 libration point on the line from the sun to the Earth-moon barycenter.[13] The cost is the magnitude of the velocity difference between the spacecraft and libration point at the end of a 2-impulse transfer. The old version of the program was used; i.e., the inclination of the parking orbit was not specified. The independent variables are the magnitude of earth departure velocity and 3 angles, which specify the position vector.

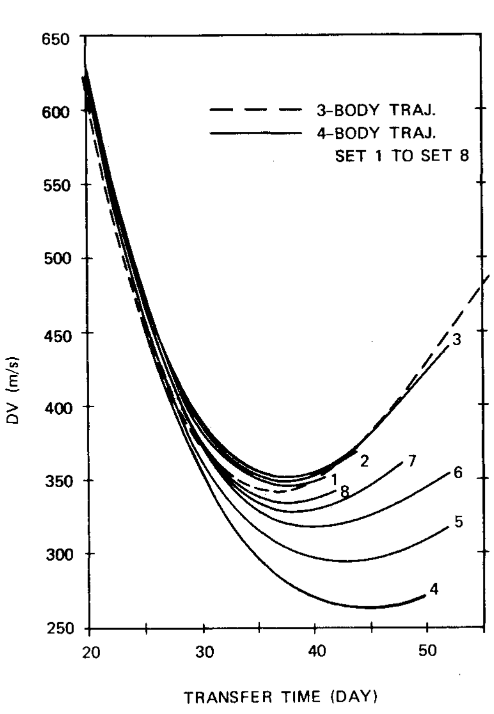

Fig. 1 ΔV vs transfer time.

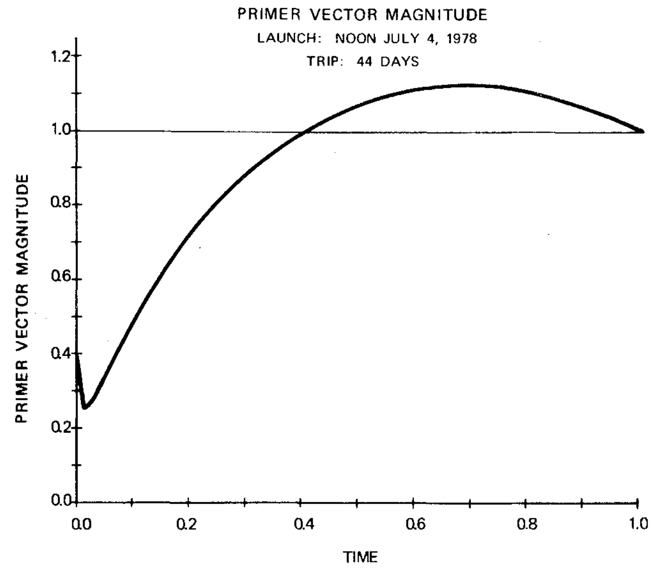

Fig. 2 Two-impulse transfer.

A family of 8 sets of trajectories was generated from noon July 1 through noon July 8, 1978. The effect of the launch date and the transfer time on terminal cost is shown in solid lines in Fig. 1. Some interesting observations of the results may be made as follows: 1) For each launch date the ΔV vs transfer time curve has a minimum value; 2) The cost of trajectories launched on July 4 is at the bottom of the set while that launched on July 3 is at the top. The difference between the 2 sets is about 80 mps at the minimum points. The spacecraft launched between July 1 through July 3 would pass between the moon and the L_1 point. The effect of the moon is to pull the spacecraft away from its course to the L_1 point. The spacecraft launched between July 4 through July 8 is assisted by the moon which passes between the spacecraft and the L_1 point. 3) The broken line curve shown in Fig. 1 is the ΔV vs transfer time curve of a family of 3-body trajectories obtained by a previous study[14] using Wilson's method. The effect of the moon is approximated by adding the mass of the moon to the mass of the Earth and increasing the initial parking orbit radius so that the velocity is the same as in a 100 nm Earth orbit. The 3-body approximation appears to be very good.

Example 2

This example shows a 2-impulse transfer from the Earth launched on noon July 4, 1978 to the L_1 libration point in 44 days. The Earth parking orbit has an inclination of 28.5°. The independent variables are the magnitude of the Earth departure velocity, the longitude of the ascending node and the angle of the spacecraft measured from the ascending node in the orbital plane. The terminal cost is to be minimized. The 2-impulse primer vector history of the converged solution is shown in Fig. 2. This transfer is not optimal for the specified transfer time.

Using the 2-impulse transfer as reference a 3-impulse transfer between the same initial and terminal position vectors is found to minimize the total cost. The 3-impulse primer vector history of the converged solution is shown in Fig. 3. The interior impulse of the converged solution is applied at about 41.9 days and is nearly at the end of the 44 day transfer. The costs are shown in Table 1.

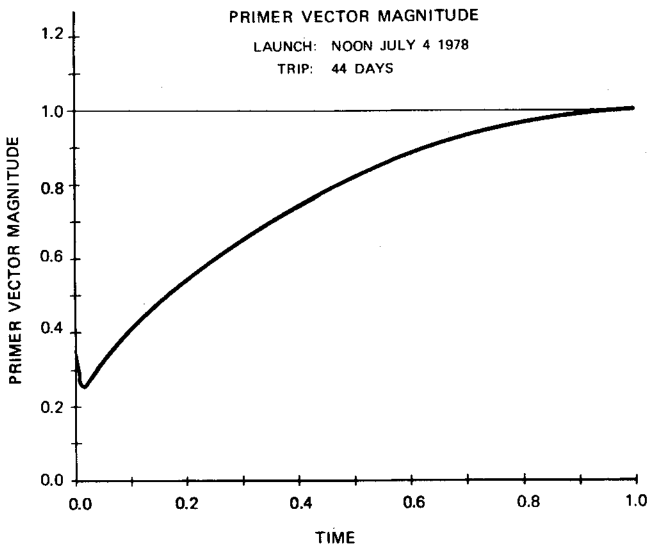

Fig. 3 Three-impulse transfer.

Table 1 Transfer costs

	2-Impulse	3-Impulse
Initial impulse	3204.20 mps	3207.35 mps
Interior impulse	—	273.44
Terminal impulse	377.80	97.64
	3582.00 mps	3578.43 mps

The 3-impulse transfer costs about 3.50 mps less than the 2-impulse transfer.

The 3-impulse transfer is difficult to generate when the 2-impulse transfer is nearly optimal. The first difficulty is to break away from the 2-impulse trajectory. The second difficulty is that the cost gradient must be reduced several orders of magnitude lower than necessary in order to satisfy the condition

$$\dot{\lambda}_m^- \cdot \lambda_m \approx \dot{\lambda}_m^+ \cdot \lambda_m$$

where λ_m is the primer vector at the interior impulse and $\dot{\lambda}_m^-$ and $\dot{\lambda}_m^+$ are the derivatives before and after the impulse.

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